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REARRANGEMENTS

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Abstract. It is shown that if $a^{(t)}=(a_1^{(t)}, a_2^{(t)}, \ldots, a_n^{(t)})$, $t=1,\ldots,m$, are nonnegative n-tuples, then the maxima of $\sum_{i=1}^n a_i^{(1)}a_i^{(2)}\cdots a_i^{(m)}$, of $\prod_{i=1}^n \min_t (a_i^{(t)})$ and of $\sum_{i=1}^n \min (a_i^{(t)})$, and the minima of $\prod_{i=1}^n (a_i^{(1)}+a_i^{(2)}+\cdots+a_i^{(m)})$, of $\prod_{i=1}^n \max_t (a_i^{(t)})$ and of $\sum_{i=1}^n \max_t (a_i^{(t)})$ are attained when the n-tuples $a^{(1)}, a^{(2)}, \ldots, a^{(m)}$ are similarly ordered. Necessary and sufficient conditions for equality are obtained in each case. An application to bounds for permanents of (0, 1)-matrices is given.

1. **Introduction.** If $x=(x_1, x_2, \ldots, x_n)$ is any real n-tuple, let $x^*=(x_1^*, x_2^*, \ldots, x_n^*)$ denote the n-tuple x rearranged in a nonincreasing order, $x_1^* \geqq x_2^* \geqq \cdots \geqq x_n^*$, and let $x'=(x_1', x_2', \ldots, x_n')$ denote the n-tuple x rearranged in a nondecreasing order, $x_1' \leqq x_2' \leqq \cdots \leqq x_n'$.

A well-known rearrangement theorem [1, Theorem 368] states that if $a=(a_1, a_2, \ldots, a_n)$ and $b=(b_1, b_2, \ldots, b_n)$ are real n-tuples, then

$$\sum_{i=1}^n a_i^* b_i' \leqq \sum_{i=1}^n a_i b_i \leqq \sum_{i=1}^n a_i' b_i'. \tag{1}$$

It is also known [6] that if $a=(a_1, a_2, \ldots, a_n)$ and $b=(b_1, b_2, \ldots, b_n)$ are nonnegative n-tuples, then

$$\prod_{i=1}^n (a_i'+b_i') \leqq \prod_{i=1}^n (a_i+b_i) \leqq \prod_{i=1}^n (a_i^*+b_i'). \tag{2}$$

Equality can hold on either side of (2) if and only if either both products contain a zero factor, or the factors of one product are just a rearrangement of the factors of the other. It is easy to show that (2) remains valid if a and b are real n-tuples satisfying $a_i+b_i \geqq 0$, $i=1,\ldots,n$, in the case of the right-hand inequality, and $a_i'+b_i' \geqq 0$, $i=1,\ldots,n$, in the case of the left-hand inequality in (2). Ruderman [7] generalized the left inequality in (2) and the right inequality in (1) to inequalities involving more then two n-tuples. In §2, I reprove Ruderman's inequalities by a direct method and obtain conditions for equality.

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Jurkat and Ryser [3] gave the following inequalities for nonnegative n-tuples $a=(a_1, a_2, \ldots, a_n)$ and $b=(b_1, b_2, \ldots, b_n)$:

$$\prod_{i=1}^{n} \min(a_i', b_i^*) \leqq \prod_{i=1}^{n} \min(a_i, b_i) \leqq \prod_{i=1}^{n} \min(a_i', b_i'). \tag{2'}$$

In the present note I generalize the right inequality in (2′) and obtain a rearrangement theorem for products of maxima. In §3 two of the inequalities are used to improve known bounds for permanents of (0, 1)-matrices.

In the concluding section it is shown that inequalities (1) and (2) are equivalent.

2. **Rearrangement inequalities.** Instead of the notation introduced in the opening paragraph, which is somewhat inconvenient when superscripts are required, we shall use the following notation. If $a^{(t)}=(a_1^{(t)}, a_2^{(t)}, \ldots, a_n^{(t)})$ is a real n-tuple, denote by $\alpha^{(t)}=(\alpha_1^{(t)}, \alpha_2^{(t)}, \ldots, \alpha_n^{(t)})$ the n-tuple $a^{(t)}$ rearranged in nondecreasing order $\alpha_1^{(t)} \leqq \alpha_2^{(t)} \leqq \cdots \leqq \alpha_n^{(t)}$.

THEOREM 1. *If $a^{(t)}=(a_1^{(t)}, a_2^{(t)}, \ldots, a_n^{(t)})$, $t=1, \ldots, m$, are nonnegative n-tuples, then*

$$\sum_{i=1}^{n} \prod_{t=1}^{m} a_i^{(t)} \leqq \sum_{i=1}^{n} \prod_{t=1}^{m} \alpha_i^{(t)}. \tag{3}$$

(*In other words, the maximum of $\sum_{i=1}^{n} a_i^{(1)} a_i^{(2)} \cdots a_i^{(m)}$ corresponds to similar ordering of the n-tuples $a^{(1)}, a^{(2)}, \ldots, a^{(m)}$.*)

Equality holds in (3) *if and only if the n-tuple*

$$\left(\prod_{t=1}^{m} a_1^{(t)}, \prod_{t=1}^{m} a_2^{(t)}, \ldots, \prod_{t=1}^{m} a_n^{(t)}\right) \tag{4}$$

is a rearrangement of the n-tuple

$$\left(\prod_{t=1}^{m} \alpha_1^{(t)}, \prod_{t=1}^{m} \alpha_2^{(t)}, \ldots, \prod_{t=1}^{m} \alpha_n^{(t)}\right). \tag{5}$$

Proof. We can assume without loss of generality that the n-tuple $a^{(m)}$ is arranged in a nondecreasing order, i.e., $a_i^{(m)}=\alpha_i^{(m)}$, $i=1, \ldots, n$. First we consider the case $n=2$, using induction on m. When $m=2$, the inequality is

$$a_1^{(1)}\alpha_1^{(2)} + a_2^{(1)}\alpha_2^{(2)} \leqq \alpha_1^{(1)}\alpha_1^{(2)} + \alpha_2^{(1)}\alpha_2^{(2)}. \tag{6}$$

This is, of course, a special case of (1). However, we prove it here in order to obtain the condition for equality. If $\alpha_1^{(1)}=a_1^{(1)}$, then (6) is an identity. If $\alpha_1^{(1)}=a_2^{(1)}$, then the difference between the right-hand side and the left-hand side of (4) is

$$(\alpha_1^{(1)}\alpha_1^{(2)} + \alpha_2^{(1)}\alpha_2^{(2)}) - (\alpha_2^{(1)}\alpha_1^{(2)} + \alpha_1^{(1)}\alpha_2^{(2)}) = (\alpha_1^{(1)} - \alpha_2^{(1)})(\alpha_1^{(2)} - \alpha_2^{(2)}) \geqq 0.$$

Thus equality holds if and only if either $\alpha_1^{(1)}=\alpha_2^{(1)}$ or $\alpha_1^{(2)}=\alpha_2^{(2)}$. Now assume that (3) holds for $n=2$ and for less than m n-tuples. We have

$$
\begin{aligned}
&\left(\prod_{t=1}^{m}\alpha_1^{(t)}+\prod_{t=1}^{m}\alpha_2^{(t)}\right)-\left(\prod_{t=1}^{m}a_1^{(t)}+\prod_{t=1}^{m}a_2^{(t)}\right)\\
&\qquad=\left(\prod_{t=1}^{m-1}\alpha_1^{(t)}-\prod_{t=1}^{m-1}a_1^{(t)}\right)\alpha_1^{(m)}+\left(\prod_{t=1}^{m-1}\alpha_2^{(t)}-\prod_{t=1}^{m-1}a_2^{(t)}\right)\alpha_2^{(m)}\\
&\qquad\geqq\left\{\left(\prod_{t=1}^{m-1}\alpha_1^{(t)}+\prod_{t=1}^{m-1}\alpha_2^{(t)}\right)-\left(\prod_{t=1}^{m-1}a_1^{(t)}+\prod_{t=1}^{m-1}a_2^{(t)}\right)\right\}\alpha_2^{(m)}\\
&\qquad\geqq 0.
\end{aligned}
\tag{7}
$$

The first inequality in (7) holds because $\prod_{t=1}^{m-1}\alpha_1^{(t)}\leqq\prod_{t=1}^{m-1}a_1^{(t)}$ and $\alpha_1^{(m)}\leqq\alpha_2^{(m)}$. The second inequality holds by the induction hypothesis. Equality can hold if and only if either $a_1^{(m)}=a_2^{(m)}=0$, or $a_1^{(m)}=a_2^{(m)}$ and the expression in the curly brackets in (7) is zero, which implies by the induction hypothesis that $(\prod_{t=1}^{m-1}a_1^{(t)}, \prod_{t=1}^{m-1}a_2^{(t)})$ is a rearrangement of $(\prod_{t=1}^{m-1}\alpha_1^{(t)}, \prod_{t=1}^{m-1}\alpha_2^{(t)})$. This proves the theorem in the case $n=2$. Now let $n>2$. If the n-tuple (4) is a rearrangement of the n-tuple (5), there is nothing to prove. Otherwise there must exist a pair of terms, $a_r^{(1)}a_r^{(2)}\cdots a_r^{(m)}$ and $a_s^{(1)}a_s^{(2)}\cdots a_s^{(m)}$, $r<s$, such that $a_r^{(t)}>a_s^{(t)}$ for some $t=t_1,\ldots,t_k$, and $a_r^{(t)}\leqq a_s^{(t)}$ for other t. Then, as we have just proved, interchanging $a_r^{(t)}$ and $a_s^{(t)}$, $t=t_1,\ldots,t_k$, does not diminish the sum of the two terms and thus also the sum $\sum_{i=1}^{n}\prod_{t=1}^{m}a_i^{(t)}$. In fact, the latter is increased by the exchange, provided the terms are not both 0. A finite number of such exchanges will transform the $\sum_{i=1}^{n}\prod_{t=1}^{m}a_i^{(t)}$ into $\sum_{i=1}^{n}\prod_{t=1}^{m}\alpha_i^{(t)}$.

Now suppose that (3) is an equality. We can assume without loss of generality that the n-tuple (4) is arranged in nondecreasing order. Suppose that $\prod_{t=1}^{m}a_k^{(t)}$ is the least positive element of (4), i.e., that

$$0=\prod_{t=1}^{m}a_1^{(t)}=\cdots=\prod_{t=1}^{m}a_{k-1}^{(t)}<\prod_{t=1}^{m}a_k^{(t)}\leqq\cdots\leqq\prod_{t=1}^{m}a_n^{(t)}.$$

Then, $a_k^{(t)}\leqq a_{k+1}^{(t)}\leqq\cdots\leqq a_n^{(t)}$, $t=1,\ldots,m$; otherwise we could increase $\sum_{i=1}^{n}\prod_{t=1}^{m}a_i^{(t)}$ by a rearrangement of one of the n-tuples. But this implies that $a_i^{(t)}\leqq\alpha_i^{(t)}$, $i=k,\ldots,n$, $t=1,\ldots,m$, and therefore

$$\prod_{t=1}^{m}a_i^{(t)}\leqq\prod_{t=1}^{m}\alpha_i^{(t)},\qquad i=1,\ldots,n.$$

Since (3) is an equality, we must have

$$\prod_{t=1}^{m}a_i^{(t)}=\prod_{t=1}^{m}\alpha_i^{(t)},\qquad i=1,\ldots,n.$$

Note that although the right inequality in (1) holds for all real n-tuples, this is not the case for inequality (3) if $m\geqq 3$. For example, if $a^{(1)}=(-1,-1)$,

$a^{(2)}=(-2,-1)$ and $a^{(3)}=(-1,-2)$, then the left-hand side of (3) is -4, while the right-hand side of (3) is -5.

THEOREM 2. *If $a^{(t)}=(a_1^{(t)},a_2^{(t)},\ldots,a_n^{(t)})$, $t=1,\ldots,m$, are nonnegative n-tuples, then*

$$\prod_{i=1}^{n}\sum_{t=1}^{m}a_i^{(t)} \geqq \prod_{i=1}^{n}\sum_{t=1}^{m}\alpha_i^{(t)}. \tag{8}$$

(*In other words, the minimum of $\prod_{i=1}(a_i^{(1)}+a_i^{(2)}+\cdots+a_i^{(m)})$ corresponds to similar ordering of the n-tuples $a^{(1)},a^{(2)},\ldots,a^{(m)}$.*) *Equality holds in* (8) *if and only if either both sides of* (8) *are* 0, *or the n-tuple $a^{(1)}+a^{(2)}+\cdots+a^{(m)}$ is a rearrangement of the n-tuple $\alpha^{(1)}+\alpha^{(2)}+\cdots+\alpha^{(m)}$* (*i.e., the factors on both sides of* (8) *are the same, except possibly for order*).

Proof. The method of proof is similar to the one used in the proof of Theorem 1. Again we assume that the n-tuple $a^{(m)}$ is arranged in a nondecreasing order. We first consider the case $n=2$ and we apply induction on m. The case $m=2$ is a special case of (2) and was proved in [6]. Now assume that (8) holds for less than m pairs. We have

$$\begin{aligned}\left(\sum_{t=1}^{m}a_1^{(t)}\right)&\left(\sum_{t=1}^{m}a_2^{(t)}\right)-\left(\sum_{t=1}^{m}\alpha_1^{(t)}\right)\left(\sum_{t=1}^{m}\alpha_2^{(t)}\right)\\ &=\left(\sum_{t=1}^{m-1}a_1^{(t)}\right)\left(\sum_{t=1}^{m-1}a_2^{(t)}\right)-\left(\sum_{t=1}^{m-1}\alpha_1^{(t)}\right)\left(\sum_{t=1}^{m-1}\alpha_2^{(t)}\right)\\ &\quad+\left(\sum_{t=1}^{m-1}a_2^{(t)}-\sum_{t=1}^{m-1}\alpha_2^{(t)}\right)\alpha_1^{(m)}+\left(\sum_{t=1}^{m}a_1^{(t)}-\sum_{t=1}^{m}\alpha_1^{(t)}\right)\alpha_2^{(m)},\end{aligned}\tag{9}$$

since $a_1^{(m)}=\alpha_1^{(m)}$ and $a_2^{(m)}=\alpha_2^{(m)}$. Now $\sum_{t=1}^{m}a_1^{(t)}-\sum_{t=1}^{m}\alpha_1^{(t)}\geqq 0$ and $\alpha_2^{(m)}\geqq\alpha_1^{(m)}$. Therefore

$$\begin{aligned}\left(\sum_{t=1}^{m-1}a_2^{(t)}-\sum_{t=1}^{m-1}\alpha_2^{(t)}\right)\alpha_1^{(m)}&+\left(\sum_{t=1}^{m}a_1^{(t)}-\sum_{t=1}^{m}\alpha_1^{(t)}\right)\alpha_2^{(m)}\\ &\geqq\left(\sum_{t=1}^{m-1}a_2^{(t)}+\sum_{t=1}^{m}a_1^{(t)}-\sum_{t=1}^{m-1}\alpha_2^{(t)}-\sum_{t=1}^{m}\alpha_1^{(t)}\right)\alpha_1^{(m)}\\ &=(-a_2^{(m)}+\alpha_2^{(m)})\alpha_1^{(m)}=0.\end{aligned}$$

Hence from (9),

$$\begin{aligned}\left(\sum_{t=1}^{m}a_1^{(t)}\right)&\left(\sum_{t=1}^{m}a_2^{(t)}\right)-\left(\sum_{t=1}^{m}\alpha_1^{(t)}\right)\left(\sum_{t=1}^{m}\alpha_2^{(t)}\right)\\ &\geqq\left(\sum_{t=1}^{m-1}a_1^{(t)}\right)\left(\sum_{t=1}^{m-1}a_2^{(t)}\right)-\left(\sum_{t=1}^{m-1}\alpha_1^{(t)}\right)\left(\sum_{t=1}^{m-1}\alpha_2^{(t)}\right)\geqq 0,\end{aligned}\tag{10}$$

by the induction hypothesis. Equality holds if and only if (10) is an equality, i.e.,

$$\left(\sum_{t=1}^{m-1}a_1^{(t)}\right)\left(\sum_{t=1}^{m-1}a_2^{(t)}\right)=\left(\sum_{t=1}^{m-1}\alpha_1^{(t)}\right)\left(\sum_{t=1}^{m-1}\alpha_2^{(t)}\right),$$

and either $\alpha_1^{(m)}=\alpha_2^{(m)}$ or $\sum_{t=1}^m a_1^{(t)}=\sum_{t=1}^m \alpha_1^{(t)}$. It is easy to see that this implies that $(\sum_{t=1}^m a_1^{(t)}, \sum_{t=1}^m a_2^{(t)})$ is either $(\sum_{t=1}^m \alpha_1^{(t)}, \sum_{t=1}^m \alpha_2^{(t)})$ or $(\sum_{t=1}^m \alpha_2^{(t)}, \sum_{t=1}^m \alpha_1^{(t)})$, unless one of $\sum_{t=1}^m a_1^{(t)}, \sum_{t=1}^m a_2^{(t)}$ is 0. This concludes the proof of the case $n=2$. Now suppose that $n>2$. If either the left-hand side of (8) is 0 (and thus the right-hand side is necessarily 0) or if the n-tuple $\sum_{t=1}^m a^{(t)}$ is a rearrangement of the n-tuple $\sum_{t=1}^m \alpha^{(t)}$, then (8) is clearly an equality. Otherwise there must exist two sums $\sum_{t=1}^m a_r^{(t)}$ and $\sum_{t=1}^m a_s^{(t)}$, $r<s$, such that $a_r^{(t)}>a_s^{(t)}$ for $t=t_1,\dots,t_k$. But then, as we have just shown, interchanging $a_r^{(t)}$ and $a_s^{(t)}$, $t=t_1,\dots,t_k$, diminishes $(\sum_{t=1}^m a_r^{(t)})(\sum_{t=1}^m a_s^{(t)})$ and thus also $\prod_{i=1}^n \sum_{t=1}^m a_i^{(t)}$. A finite number of such interchanges will transform the product $\prod_{i=1}^n \sum_{t=1}^m a_i^{(t)}$ into $\prod_{i=1}^n \sum_{t=1}^m \alpha_i^{(t)}$. The conditions for equality in (8) follow immediately.

Our next theorem holds for nonnegative n-tuples but the inequality is completely trivial if any of the n-tuples has in fact a zero coordinate.

THEOREM 3. *If* $a^{(t)}=(a_1^{(t)},\dots,a_n^{(t)})$, $t=1,\dots,m$, *are positive n-tuples, then*

$$\prod_{i=1}^n \min_t (a_i^{(t)}) \leqq \prod_{i=1}^n \min_t (\alpha_i^{(t)}), \tag{11}$$

with equality if and only if the n-tuple $(\min_t (a_1^{(t)}), \min_t (a_2^{(t)}),\dots,\min_t (a_n^{(t)}))$ *is a rearrangement of the n-tuple* $(\min_t (\alpha_1^{(t)}), \min_t (\alpha_2^{(t)}),\dots,\min_t (\alpha_n^{(t)}))$.

Proof. We can assume without loss of generality that $a_i^{(m)}=\alpha_i^{(m)}$, $i=1,\dots,n$. As in the proofs of the preceding two theorems, the crux of the proof is to establish inequality (11) for the case of two n-tuples. Let $m=2$ and use induction on n. If $n=2$, then the inequality is

$$\min (a_1^{(1)}, \alpha_1^{(2)})\cdot \min (a_2^{(1)}, \alpha_2^{(2)}) \leqq \min (\alpha_1^{(1)}, \alpha_1^{(2)})\cdot \min (\alpha_2^{(1)}, \alpha_2^{(2)}). \tag{12}$$

We can assume that $\alpha_1^{(2)}\leqq\alpha_1^{(1)}$. If $\alpha_1^{(1)}=a_1^{(1)}$ then (12) is an identity. It remains therefore to prove the case $\alpha_1^{(1)}=a_2^{(1)}$, i.e., to show that

$$\min (\alpha_2^{(1)}, \alpha_1^{(2)})\cdot \min (\alpha_1^{(1)}, \alpha_2^{(2)}) \leqq \min (\alpha_1^{(1)}, \alpha_1^{(2)})\cdot \min (\alpha_2^{(1)}, \alpha_2^{(2)}). \tag{13}$$

If $\alpha_2^{(2)}\leqq\alpha_1^{(1)}$, then (13) is an equality. If $\alpha_2^{(2)}>\alpha_1^{(1)}$, then the left side of (13) is $\alpha_1^{(2)}\alpha_1^{(1)}$ while the right side is the product of $\alpha_1^{(2)}$ and a number greater than $\alpha_1^{(1)}$, unless $\alpha_2^{(1)}=\alpha_1^{(1)}$ in which case (13) again is an equality. This proves (11) and the necessity of the condition of equality in the case $m=n=2$. Now we assume for the purpose of induction that the theorem holds for $(n-1)$-tuples when $m=2$. We have to prove that

$$\prod_{i=1}^n \min (a_i^{(1)}, \alpha_i^{(2)}) \leqq \prod_{i=1}^n \min (\alpha_i^{(1)}, \alpha_i^{(2)}), \tag{14}$$

where we can assume without loss of generality that $\alpha_1^{(2)}\leqq a_1^{(1)}$. We have

$$\begin{aligned}\prod_{i=1}^n \min (a_i^{(1)}, \alpha_i^{(2)}) &= \alpha_1^{(2)} \prod_{i=2}^n \min (a_i^{(1)}, \alpha_i^{(2)})\\ &\leqq \alpha_1^{(2)} \prod_{i=2}^n \min (\tilde{a}_i^{(1)}, \alpha_i^{(1)}),\end{aligned} \tag{15}$$

where $\tilde{a}_2^{(1)}, \ldots, \tilde{a}_n^{(1)}$ are the numbers $a_2^{(1)}, \ldots, a_n^{(1)}$ arranged in a nondecreasing order. But

$$\tilde{a}_i^{(1)} \leqq \alpha_i^{(1)}, \qquad i = 2, \ldots, n, \tag{16}$$

and therefore by (15) and (16),

$$\prod_{i=1}^n \min(a_i^{(1)}, \alpha_i^{(2)}) \leqq \alpha_1^{(2)} \prod_{i=2}^n \min(\alpha_i^{(1)}, \alpha_i^{(2)}) = \prod_{i=1}^n \min(\alpha_i^{(1)}, \alpha_i^{(2)}).$$

Equality can hold only if $\tilde{a}_i^{(1)} = \alpha_i^{(1)}$, $i = 2, \ldots, n$, i.e., only if $a_1^{(1)} = \alpha_1^{(1)}$, and, by the induction hypothesis, the $(n-1)$-tuple $(\min(a_2^{(1)}, \alpha_2^{(2)}), \ldots, \min(a_n^{(1)}, \alpha_n^{(2)}))$ is a rearrangement of the $(n-1)$-tuple $(\min(\alpha_2^{(1)}, \alpha_2^{(2)}), \ldots, \min(\alpha_n^{(1)}, \alpha_n^{(2)}))$. But then $\min(a_1^{(1)}, \alpha_1^{(2)}) = \min(\alpha_1^{(1)}, \alpha_1^{(2)})$ and therefore the n-tuple $(\min(a_1^{(1)}, \alpha_1^{(2)}), \min(a_2^{(1)}, \alpha_2^{(2)}), \ldots, \min(a_n^{(1)}, \alpha_n^{(2)}))$ is a rearrangement of the n-tuple $(\min(\alpha_1^{(1)}, \alpha_1^{(2)}), \min(\alpha_2^{(1)}, \alpha_2^{(2)}), \ldots, \min(\alpha_n^{(1)}, \alpha_n^{(2)}))$. This concludes the proof of the case $m = 2$. The argument extending the result to any number m of n-tuples is almost a repetition of the corresponding arguments used in the proofs of Theorems 1 and 2.

THEOREM 4. *If* $a^{(t)} = (a_1^{(t)}, \ldots, a_n^{(t)})$, $t = 1, \ldots, m$, *are nonnegative n-tuples, then*

$$\prod_{i=1}^n \max_t (a_i^{(t)}) \geqq \prod_{i=1}^n \max_t (\alpha_i^{(t)}). \tag{17}$$

Equality holds in (17) *if and only if either both sides of* (17) *are* 0, *or if the n-tuple* $(\max_t (a_1^{(t)}), \max_t (a_2^{(t)}), \ldots, \max_t (a_n^{(t)}))$ *is a rearrangement of the n-tuple* $(\max_t (\alpha_1^{(t)}), \max_t (\alpha_2^{(t)}), \ldots, \max_t (\alpha_n^{(t)}))$.

The proof is a virtual repetition, mutatis mutandis, of the proof of Theorem 3 and will be omitted.

THEOREM 5. *If* $a^{(t)} = (a_1^{(t)}, \ldots, a_n^{(t)})$, $t = 1, \ldots, n$, *are real n-tuples, then*

$$\sum_{i=1}^n \min_t (a_i^{(t)}) \leqq \sum_{i=1}^n \min_t (\alpha_i^{(t)}) \tag{18}$$

and

$$\sum_{i=1}^n \max_t (a_i^{(t)}) \geqq \sum_{i=1}^n \max_t (\alpha_i^{(t)}). \tag{19}$$

Equality can hold in (18) *if and only if the n-tuple* $(\min_t (a_1^{(t)}), \min_t (a_2^{(t)}), \ldots, \min_t (a_n^{(t)}))$ *is a rearrangement of the n-tuple* $(\min_t (\alpha_1^{(t)}), \min_t (\alpha_2^{(t)}), \ldots, \min_t (\alpha_n^{(t)}))$. *Equality can hold in* (19) *if and only if the n-tuple* $(\max_t (a_1^{(t)}), \max_t (a_2^{(t)}), \ldots, \max_t (a_n^{(t)}))$ *is a rearrangement of the n-tuple* $(\max_t (\alpha_1^{(t)}), \max_t (\alpha_2^{(t)}), \ldots, \max_t (\alpha_n^{(t)}))$.

Proof. Set $b_i^{(t)} = e^{a_i^{(t)}}$, $\beta_i^{(t)} = e^{\alpha_i^{(t)}}$, $i = 1, \ldots, n$, $t = 1, \ldots, m$. Inequalities (18) and (19) then follow from Theorems 3 and 4.

3. **Applications to permanental inequalities.** Let $A=(a_{ij})$ be an n-square (0, 1)-matrix, i.e., a matrix all of whose entries are 0 and 1. Let $r_i=\sum_{j=1}^{n} a_{ij}$, $i=1,\ldots,n$. Jurkat and Ryser [2] (see also [5]) have shown that

$$\prod_{i=1}^{n} \max(r_i+1-i, 0) \leqq \operatorname{per}(A) \leqq \prod_{i=1}^{n} \min(r_i, i), \tag{20}$$

where per (A) denotes the permanent of A, i.e.,

$$\operatorname{per}(A) = \sum_{\sigma\in S_n} \prod_{i=1}^{n} a_{i,\sigma(i)}.$$

The permanent of a matrix is invariant under permutations of rows of the matrix. However, the bounds in (20) are not invariant and can be improved by a judicious permutation of the rows of A.

LEMMA 1. *If $r_1,\ldots,r_n$ are integers, $0\leqq r_i\leqq n$, $i=1,\ldots,n$, then*

$$\prod_{i=1}^{n} \max(r_i+1-i, 0) \leqq \prod_{i=1}^{n} \max(r'_i+1-i, 0). \tag{21}$$

Proof. If the left-hand side of (21) is positive, then (21) follows from the right inequality in (2) by setting $a_i=r_i$ and $b_i=1-i$, $i=1,\ldots,n$. It remains to show that if the right-hand side of (21) is 0, then the left-hand side is necessarily 0. Suppose that $r'_s+1-s\leqq 0$, for some s, $1\leqq s\leqq n$. Then we assert that we must also have $r_t+1-t\leqq 0$ for some t, $s\leqq t\leqq n$. For, if $r_s+1-s>0$, then $r_s>r'_s$, and therefore there exists an integer t, $t>s$, such that $r_t\leqq r'_s$. It follows that $r_t+1-s\leqq 0$ and thus, a fortiori, $r_t+1-t<0$. Hence both sides of (21) are 0.

LEMMA 2. *If $r_1,\ldots,r_n$ are integers, $0\leqq r_i\leqq n$, $i=1,\ldots,n$, then*

$$\prod_{i=1}^{n} \min(r_i, i) \geqq \prod_{i=1}^{n} \min(r^*_i, i). \tag{22}$$

Proof. Set $a_i=i$ and $b_i=r_i$, $i=1,\ldots,n$, in the left inequality in (2′).

It follows from Lemmas 1 and 2, that the following theorem sharpens the result of Jurkat and Ryser.

THEOREM 6. *If A is an n-square* (0, 1)-*matrix with row sums $r_1,\ldots,r_n$ then*

$$\prod_{i=1}^{n} \max(r'_i+1-i, 0) \leqq \operatorname{per}(A) \leqq \prod_{i=1}^{n} \min(r^*_i, i).$$

4. **The equivalence of inequalities (1) and (2).** In this section we show that for positive n-tuples the inequalities (1) and (2) are equivalent. The author is indebted to David London and to Victor J. Mizel for their assistance in reaching this conclusion.

We show that the right inequality in (2) implies the right inequality in (1). Set in (2): $a_i=1/c_i$ and $b_i=\varepsilon d_i$, $i=1,\ldots,n$, where ε is any positive real number. If we then multiply both sides by $\prod_{i=1}^{n} c_i$, the inequality reads

$$\prod_{i=1}^{n} (1+\varepsilon c_i d_i) \leqq \prod_{i=1}^{n} (1+\varepsilon c_i' d_i'), \tag{23}$$

since $a_i^* = 1/c_i'$, $i=1, \ldots, n$. Thus

$$1+\sum_{r=1}^{n} \varepsilon^r E_r \leqq 1+\sum_{r=1}^{n} \varepsilon^r E_r', \tag{24}$$

where E_r denotes the rth elementary symmetric function of $c_1 d_1, \ldots, c_n d_n$ and E' the rth elementary symmetric function of $c_1' d_1', \ldots, c_n' d_n'$. It follows from (24) that

$$\sum_{i=1}^{n} c_i d_i + \varepsilon\left(\sum_{r=2}^{n} \varepsilon^{r-2} E_r\right) \leqq \sum_{i=1}^{n} c_i' d_i' + \varepsilon\left(\sum_{r=2}^{n} \varepsilon^{r-2} E_r'\right),$$

and since the inequality holds for all positive ε, we must have

$$\sum_{i=1}^{n} c_i d_i \leqq \sum_{i=1}^{n} c_i' d_i'. \tag{25}$$

We now show that (25) implies (23), and thus for positive n-tuples it also implies the right inequality in (2). If (25) is a strict inequality then the above reasoning can be reversed for sufficiently small positive ε. If (25) is an equality, then by Theorem 1, the n-tuple $(c_1 d_1, \ldots, c_n d_n)$ is a rearrangement of the n-tuple $(c_1' d_1', \ldots, c_n' d_n')$. Hence $E_r = E_r'$, $r=1, \ldots, n$, for any ε, and (24), and thus also (23), is an inequality for any ε.

We can prove in a similar manner that the left inequalities in (1) and (2) are equivalent for positive n-tuples.

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